Chapter 5

APPLICATIONS OF DERIVATIVES

We live in a world that is always in flux. Sir Isaac Newton's name for calculus was "the method of fluxions." He recognized in the 17th century, as you probably recognize today, that understanding change is important. Newton was what we might call a "mathematical physicist." He developed his method of fluxions as a means to better understand the natural world, including motion and gravity. But change is not limited to the natural world, and since Newton's time, the use of calculus has spread to include applications in the social sciences. Psychology, business, and economics are just a few of the areas in which calculus has been an effective problem-solving tool. As we shall see in this chapter, anywhere that functions can be used as models, the derivative is certain to be meaningful and useful.

CHAPTER EXPECTATIONS In this chapter, you will

- justify the rules for determining derivatives, **Section 5.1**
- determine derivatives of polynomial and rational functions using rules for determining derivatives, **Section 5.1**
- solve problems of rates of change, **Section 5.1**
- determine second derivatives, **Section 5.2**
- solve related-rates problems, **Section 5.3**
- determine key features of the graph of a function, **Section 5.4**
- determine key features of a mathematical model, pose questions, and answer them by analyzing mathematical models, **Section 5.5**
- solve optimization problems, **Section 5.5, 5.6, Career Link**
- communicate findings clearly and concisely, **Section 5.6**

Review of Prerequisite Skills

Now that you have developed your understanding of derivatives and differentiation techniques in Chapter 4, we will consider a variety of applications of derivatives. The following skills will help you in your work in this chapter:

- Graphing polynomial and simple rational functions
- Drawing circles, ellipses, and hyperbolas in both standard position and when translated
- Solving polynomial equations
- Finding the equations of tangents and normals
- You should also be familiar with the following formulas:

Circle:	$C = 2\pi r$, $A = \pi r^2$
Right Circular Cylinder:	$S = 2\pi rh + 2\pi r^2$, $V = \pi r^2 h$
Sphere:	$S = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$
Right Circular Cone:	$V = \frac{1}{3}\pi r^2 h$

Exercise

1. Sketch the graph of each function.
   a. $2x + 3y - 6 = 0$ b. $3x - 4y = 12$ c. $y = \sqrt{x}$
   d. $y = \sqrt{x - 2}$ e. $y = x^2 - 4$ f. $y = -x^2 + 9$

2. Draw each of the following circles:
   a. $x^2 + y^2 = 9$ b. $(x - 2)^2 + (y - 3)^2 = 9$
   c. $(x + 4)^2 + (y - 1)^2 = 49$

3. Draw each of the following ellipses:
   a. $4x^2 + 9y^2 = 36$ b. $x^2 + 4y^2 = 100$ c. $\frac{(x + 4)^2}{49} + \frac{(y - 1)^2}{4} = 1$

4. Draw each hyperbola defined by the following equations:
   a. $xy = 4$ b. $4x^2 - 9y^2 = 36$ c. $x^2 - 4y^2 = -100$

5. Solve $x, t \in R$.
   a. $3(x - 2) + 2(x - 1) - 6 = 0$ b. $\frac{1}{3}(x - 2) + \frac{2}{5}(x + 3) = \frac{x - 5}{2}$
   c. $t^2 - 4t + 3 = 0$ d. $2t^2 - 5t - 3 = 0$

e. $\frac{6}{t} + \frac{t}{2} = 4$

f. $x^3 + 2x^2 - 3x = 0$

g. $x^3 - 8x^2 + 16x = 0$

h. $4t^3 + 12t^2 - t - 3 = 0$

i. $4t^4 - 13t^2 + 9 = 0$

6. Solve each inequality, $x \in R$.

a. $3x - 2 > 7$ b. $x(x - 3) > 0$ c. $-x^2 + 4x > 0$

7. Find the area of the figure described. Leave your answers in terms of π.

a. Square: perimeter 20 cm

b. Rectangle: length 8 cm, width 6 cm

c. Circle: radius 7 cm

d. Circle: circumference 12π cm

8. Two measures of a right circular cylinder are given. Find the two remaining measures.

	Radius r	Height h	Surface Area $S = 2\pi rh + 2\pi r^2$	Volume $V = \pi r^2 h$
a.	4 cm	3 cm		
b.	4 cm			96π cm^3
c.		6 cm		216π cm^3
d.	5 cm		120π cm^2	

9. One measure of a sphere is given. Find the two remaining measures.

	Radius r	Volume $V = \frac{4}{3}\pi r^3$	Surface Area $S = 4\pi r^2$
a.	9 cm		
b.	3 cm		
c.		36π cm^3	
d.			1000π cm^2

10. Two measures of a right circular cone are given. Find the remaining measure.

	r	h	$V = \frac{1}{3}\pi r^2 h$
a.	4 cm	3 cm	
b.	3 cm		27π cm^3
c.		4 cm	27π cm^3

11. Find the total surface area and volume of a cube with each of the following dimensions:

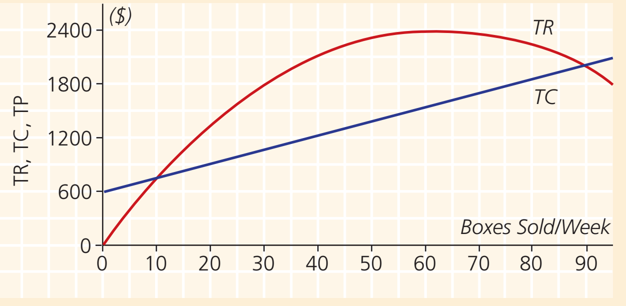
a. 3 cm b. $\sqrt{5}$ cm c. $2\sqrt{3}$ cm d. $2k$ cm

Career Link investigate

CHAPTER 5: MAXIMIZING PROFITS

We live in a world that demands we determine the best, the worst, the maximum, and the minimum. Through mathematical modelling, calculus can be used to establish optimum operating conditions for processes that seem to have competing variables. For example, minimizing transportation costs for a delivery truck would seem to require the driver to travel as fast as possible to reduce hourly wages. Higher rates of speed, however, increase the cost of gas consumption. With calculus, an optimal speed can be established that minimizes the total cost of driving the delivery vehicle considering both gas consumption and hourly wages. In this chapter, calculus tools will be utilized in realistic contexts to solve optimization problems from business applications (e.g., minimizing cost) to psychology (e.g., maximizing learning).

Case Study — Entrepreneurship



In the last ten years, the Canadian economy has seen a dramatic increase in the number of small businesses. An ability to interpret the marginal profit on graphs, a calculus concept, will help an entrepreneur to make good business decisions. A person with an old family recipe for gourmet chocolates decides to open her own business. Her weekly total revenue (*TR*) and total cost (*TC*) curves are plotted on the set of axes below.

DISCUSSION QUESTIONS

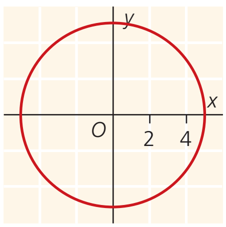
Make a rough sketch of the graph in your notes and answer the following questions:

1. What range of sales would keep the company profitable? What do we call these values?
2. Superimpose the total profit (*TP*) curve over the *TR* and *TC* curves. What would the sales level have to be for maximum profits to occur? Estimate the slopes on the *TR* and *TC* curves at this level of sales. Should they be the same? Why or why not?
3. On a set of separate axes, draw a rough sketch of the marginal profit $\left(MP = \frac{dTP}{dx}\right)$, the extra profit earned by selling one more box of chocolates. What can you say about the marginal profit as the level of sales progress from just less than the maximum to the maximum to just above the maximum? Does this make sense? Explain.

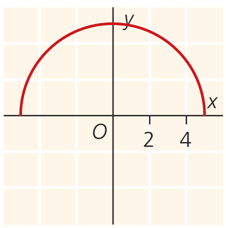
Section 5.1 — Implicit Differentiation

In previous chapters, most functions were written in the form $y = f(x)$, in which y is defined **explicitly** as a function of x. Examples of functions that are defined explicitly include $y = x^3 - 4x$ and $y = \frac{7}{x^2 + 1}$. However, functions can also be defined **implicitly** by relations such as the circle $x^2 + y^2 = 25$.

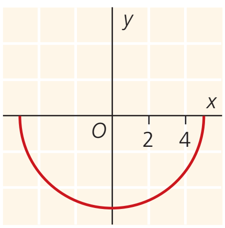
Since there are x-values that correspond to two y-values, y is not a function of x on the entire circle. Solving for y gives $y = \pm\sqrt{25 - x^2}$, where $y = \sqrt{25 - x^2}$ represents the upper semicircle and $y = -\sqrt{25 - x^2}$ is the lower semicircle. The given relation defines two different functions of x.



$x^2 + y^2 = 25$



$y = \sqrt{25 - x^2}$



$y = -\sqrt{25 - x^2}$

Consider the problem of finding the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$. Since this point lies on the lower semicircle, we could differentiate the function $y = -\sqrt{25 - x^2}$ and substitute $x = 3$. An alternative, which avoids having to solve for y explicitly in terms of x, is to use the method of **implicit differentiation**. Example 1 illustrates this method.

EXAMPLE 1

a. If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

b. Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution

a. Differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

To find $\frac{d}{dx}(y^2)$, use the Chain Rule.

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \bullet \frac{dy}{dx}$$

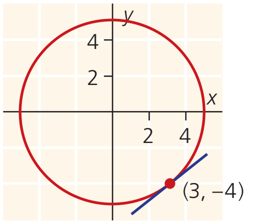
$$= 2y\frac{dy}{dx}$$

Therefore, $\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$

$$2x + 2y\frac{dy}{dx} = 0$$

and
$$\frac{dy}{dx} = -\frac{x}{y}.$$

The derivative in part **a** depends on both x and y. At the point $(3, -4)$, $x = 3$ and $y = -4$.
The slope of the tangent line to $x^2 + y^2 = 25$ at $(3, -4)$ is $\frac{dy}{dx} = -\frac{3}{-4} = \frac{3}{4}$.



In Example 1, the derivative could be found either by implicit differentiation or by solving for y in terms of x and using the techniques introduced earlier in the text. There are many situations in which solving for y in terms of x is very difficult, and in some cases impossible. In such cases, implicit differentiation is the only method available to us.

EXAMPLE 2

Find $\frac{dy}{dx}$ for $2xy - y^3 = 4$.

Solution

We differentiate both sides of the equation with respect to x as follows:

$$\frac{d}{dx}(2xy) - \frac{d}{dx}(y^3) = \frac{d}{dx}(4).$$

Use the Product Rule to differentiate the first term and the Chain Rule for the second.

$$\left(\frac{d}{dx}(2x)\right)y + 2x\frac{dy}{dx} - \frac{d}{dy}(y^3) \bullet \frac{dy}{dx} = \frac{d}{dx}(4)$$

$$2y + 2x\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$(2x - 3y^2)\frac{dy}{dx} = -2y$$

$$\frac{dy}{dx} = -\frac{2y}{2x - 3y^2}$$

Procedure for Implicit Differentiation

Suppose an equation defines y implicitly as a differentiable function of x. To find $\frac{dy}{dx}$:

Step 1. **Differentiate both sides of the equation with respect to x. Remember to use the Chain Rule when differentiating terms containing y.**

Step 2. **Solve for $\frac{dy}{dx}$.**

Note that implicit differentiation leads to a derivative whenever the derivative does not have a zero in the denominator. The derivative expression usually includes terms in both x and y.

EXAMPLE 3 Find the slope of the tangent to the ellipse $x^2 + 4y^2 = 25$ at the point $(-3, 2)$. Illustrate the tangent on the graph of the ellipse.

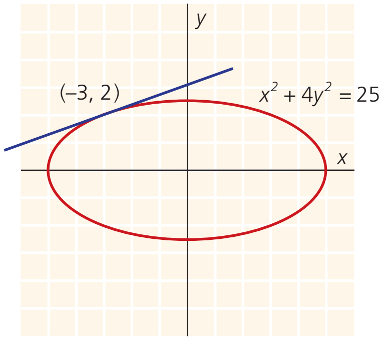
Solution

Differentiating implicitly with respect to x, we obtain

$$2x + 4\left(2y\frac{dy}{dx}\right) = 0 \text{ or } 2x + 8y\frac{dy}{dx} = 0.$$

At point $(-3, 2)$,

$$\begin{aligned} 2(-3) + 8(2)\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{6}{16} \\ &= \frac{3}{8}. \end{aligned}$$



Therefore, the slope of the tangent to the ellipse at $(-3, 2)$ is $\frac{3}{8}$.

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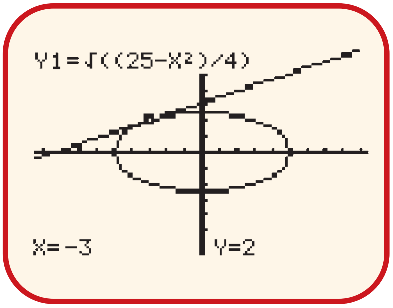
Using your graphing calculator:

Step 1. To graph the ellipse, write $x^2 + 4y^2 = 25$ as

$$\begin{aligned} 4y^2 &= 25 - x^2 \\ y^2 &= \frac{25 - x^2}{4} \\ y &= \pm\sqrt{\frac{25 - x^2}{4}}. \end{aligned}$$

First, graph $y_1 = \sqrt{\frac{25 - x^2}{4}}$ as $\sqrt{((25 - x^2) \div 4)}$.



Step 2. To graph $y = -\sqrt{\frac{25 - x^2}{4}}$, select $y_2 = -$ and then select the **VARS** function.

Then select Y-VARS and press **ENTER** twice.

Then select **GRAPH**.

Use $x_{min} = -9.4$, $x_{max} = 9.4$, $y_{min} = -6.4$, and $y_{max} = 6.4$ as your window.

Step 3. To graph the tangent at $(-3, 2)$ from the GRAPH window, select **2nd** **PRGM** (DRAW) to get the DRAW function. Select **5:Tangent(**. Move the cursor to $(-3, 2)$ and press **ENTER** to get the tangent.

Exercise 5.1

Part A

Communication

1. State the Chain Rule. Outline a procedure for implicit differentiation.

Knowledge/Understanding

2. Find $\frac{dy}{dx}$ for each of the following in terms of x and y using implicit differentiation.

a. $x^2 + y^2 = 36$
b. $y^2 = x^2 - 16$
c. $15y^2 = 2x^3$
d. $3xy^2 + y^3 = 8$
e. $5y^4 = x^3 + 13$
f. $9x^2 - 16y^2 = -144$
g. $\frac{x^2}{16} + \frac{3y^2}{13} = 1$
h. $3x^2 + 4xy^3 = 9$
i. $x^2 + y^2 + 5y = 10$
j. $x^3 + y^3 = 6xy$
k. $x^3y^3 = 144$
l. $x = y + y^5$
m. $xy^3 - x^3y = 2$
n. $\sqrt{x} + \sqrt{y} = 5$
o. $(x + y)^2 = x^2 + y^2$

3. For each curve, find the equation of the tangent at the given point.

a. $x^2 + y^2 = 13$ at $(2, -3)$
b. $x^2 + 4y^2 = 100$ at $(-8, 3)$
c. $\frac{x^2}{25} - \frac{y^2}{36} = -1$ at $(5\sqrt{3}, -12)$
d. $\frac{x^2}{81} - \frac{5y^2}{162} = 1$ at $(-11, -4)$

Part B

4. At what point is the tangent to the curve $x + y^2 = 1$ parallel to the line $x + 2y = 0$?

5. The equation $5x^2 - 6xy + 5y^2 = 16$ represents an ellipse.

a. Find $\frac{dy}{dx}$ at $(1, -1)$.

b. Find two points on the ellipse at which the tangent is horizontal.

Application

6. Find the slope of the tangent to the ellipse $5x^2 + y^2 = 21$ at point $A(-2, -1)$.

7. Find the equation of the normal at $(2, 3)$ to the curve $x^3 + y^3 - 3xy = 17$ at point $(2, 3)$.

8. Find the equation of the normal to $y^2 = \frac{x^3}{2 - x}$ at point $(1, -1)$.

9. The equation $4x^2y - 3y = x^3$ implicitly defines y as a function of x.

a. Use implicit differentiation to find $\frac{dy}{dx}$.

b. Write y as an explicit function of x and compute $\frac{dy}{dx}$ directly. Show that the results of parts **a** and **b** are equivalent.

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10. Graph each relation using a graphing calculator or a computer. For each graph, decide the number of tangents that exist at $x = 1$.

a. $y = \sqrt{3 - x}$

b. $y = -\sqrt{5 - x}$

c. $y = x^7 - x$

d. $x^3 + 4x^2 + (x - 4)y^2 = 0$ (This curve is known as the *strophoid*.)

11. Show that for the relation $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 10, x \neq y \neq 0, \frac{dy}{dx} = \frac{y}{x}$.

Part C

12. Find the equations of the lines that are tangent to the ellipse $x^2 + 4y^2 = 16$ and that also pass through the point $(4, 6)$.

13. The angle between two intersecting curves is defined as the angle between their tangents at the point of intersection. If this angle is 90º, the two curves are said to be *orthogonal*. Prove that the curves defined by $x^2 - y^2 = k$ and $xy = p$ intersect orthogonally for all values of the constants k and p. Illustrate your proof with a sketch.

14. Let l be any tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{k}$. Show that the sum of the intercepts of l is k.

15. Two circles of radius $3\sqrt{2}$ are tangent to the graph $y^2 = 4x$ at point $(1, 2)$. Find the equations of these two circles.

Section 5.2 — Higher-Order Derivatives, Velocity, and Acceleration

Derivatives arise in the study of motion. The velocity of a car represents the rate of change of distance with respect to time.

Up to this point, we have developed the rules of differentiation and learned how to interpret them at a point on a curve. We can now extend the applications of differentiation to higher-order derivatives. This will allow us to discuss the application of the first and second derivatives to rates of change as an object moves in a straight line, either vertically or horizontally, such as a space shuttle taking off into space or a car moving along a road.

Higher-Order Derivatives

The derivative of $10x^4$ with respect to x is $40x^3$. If we differentiate $40x^3$, we obtain $120x^2$. This new function is called the second derivative of $10x^4$. For $y = 2x^3 - 5x^2$, the first derivative is $\frac{dy}{dx} = 6x^2 - 10x$ and the second derivative is $\frac{d^2y}{dx^2} = 12x - 10$.

Note the location of the superscripts in the second derivative. The reason for this choice of notation is that the second derivative is the derivative of the first derivative; that is, we write $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

Other notations used to represent first and second derivatives of $y = f(x)$ are

$$\frac{dy}{dx} = f'(x) = y' \text{ and } \frac{d^2y}{dx^2} = f''(x) = y''$$

EXAMPLE 1 Find the second derivative of $f(x) = \frac{x}{1 + x}$ at $x = 1$.

Solution

Differentiate $f(x) = \frac{x}{1 + x}$ using the Quotient Rule.

$$\begin{aligned} f'(x) &= \frac{(1)(1 + x) - x(1)}{(1 + x)^2} \\ &= \frac{1 + x - x}{(1 + x)^2} \\ &= \frac{1}{(1 + x)^2} \\ &= (1 + x)^{-2} \end{aligned}$$

Differentiating again to determine the second derivative,

$$f''(x) = -2(1 + x)^{-3}(1) \quad \text{(Power of a Function Rule)}$$
$$= \frac{-2}{(1 + x)^3}.$$

At $x = 1$, $f''(1) = \frac{-2}{(1 + 1)^3}$

$$= \frac{-2}{8}$$
$$= -\frac{1}{4}.$$

Velocity and Acceleration — Motion on a Straight Line

One reason for introducing the derivative is the need to calculate rates of change. Consider the motion of an object along a straight line. Examples are a car moving along a straight section of road, a ball being dropped from the top of a building, and a rocket in the early stages of flight.

When studying motion along a line, we assume the object is moving along a coordinate line, which gives us an origin of reference and positive and negative directions. The position of the object on the line relative to the origin is a function of time, t, and is denoted by $s(t)$. The rate of change of $s(t)$ with respect to time is the object's **velocity**, $v(t)$, and the rate of change of the velocity with respect to time is its **acceleration**, $a(t)$. The absolute value of the velocity is called **speed**.

Motion on a Straight Line

An object that moves along a straight line with its position determined by the function $s(t)$ has a velocity of $v(t) = s'(t)$ and *acceleration* of $a(t) = v'(t) = s''(t)$.

In Leibniz notation,

$$v = \frac{ds}{dt} \text{ and } a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

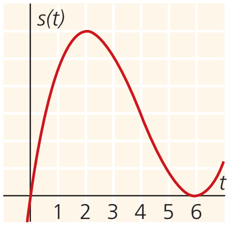
The *speed* of an object is $|v(t)|$.

The dimensions of velocity are length divided by time; typical units are m/s. The dimensions of acceleration are length divided by (time)^2; typical units are m/s^2.

If $v(t) > 0$, the object is moving to the right, and if $v(t) < 0$, it is moving to the left. If $v(t) = 0$, the object is *stationary*, or at rest. The object is *accelerating* if the product of $a(t)$ and $v(t)$ is positive and *decelerating* if the product is negative.

EXAMPLE 2

An object is moving along a straight line. Its distance, $s(t)$, to the right of a fixed point is given by the graph shown.

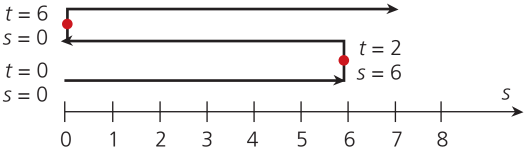


When is the object moving to the right, when is it moving to the left, and when is it at rest?

Solution

The object is moving to the right whenever $s(t)$ is increasing, or $v(t) > 0$.
From the graph, $s(t)$ is increasing for $0 < t < 2$ and for $t > 6$.
For $2 < t < 6$, the value of $s(t)$ is decreasing, so the object is moving to the left.
At $t = 6$, the direction of motion of the object changes from left to right, so the object is stationary at $t = 6$.

The motion of the object along the distance lines can be illustrated by the following diagram:



EXAMPLE 3

The position of an object moving on a line is given by $s(t) = 6t^2 - t^3$, $t \geq 0$, where s is in metres and t is in seconds.

a. Find the object's velocity and acceleration at $t = 2$.

b. At what time(s) is the object at rest?

c. In which direction is the object moving at $t = 5$?

d. When is the object moving in a positive direction?

e. When does the object return to its initial position?

Solution

a. The velocity at time t is

$$v(t) = s'(t) = 12t - 3t^2.$$

$$\text{At} \quad t = 2, \; v(2) = 12(2) - 3(2)^2 = 12.$$

The acceleration at time t is

$$\begin{aligned} a(t) = v'(t) &= s''(t) \\ &= 12 - 6t. \end{aligned}$$

$$\text{At} \quad t = 2, \; a(2) = 12 - 6(2) = 0.$$

At $t = 2$, the velocity is 12 m/s and the acceleration is 0.

We note that at $t = 2$, the object is neither speeding up nor slowing down.

b. The object is at rest when the velocity is 0, that is, $v(t) = 0$.

$$12t - 3t^2 = 0$$
$$3t(4 - t) = 0$$
$$t = 0 \text{ or } t = 4$$

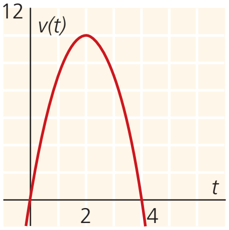
The object is at rest at $t = 0$ s and at $t = 4$ s.

c. $v(5) = 12(5) - 3(5^2)$
$= -15$

The object is moving in a negative direction at $t = 5$.

d. The object moves in a positive direction when $v(t) > 0$;
that is, when $v(t) = 12t - 3t^2 > 0$
$t^2 - 4t < 0,$ (Divide by -3)
therefore, $0 < t < 4$.

The graph of the velocity function is a parabola opening downward, as shown.



From the graph, we conclude that $v(t) > 0$ for $0 < t < 4$.

The object is moving to the right during the interval $0 < t < 4$.

e. At $t = 0$, $s(0) = 0$. Therefore, the object's initial position is at 0.

To find other times when the object is at this point, we solve $s(t) = 0$.

$$6t^2 - t^3 = 0$$
$$t^2(6 - t) = 0$$
$$t = 0 \quad \text{or} \quad t = 6$$

The object returns to its initial position after 6 s.

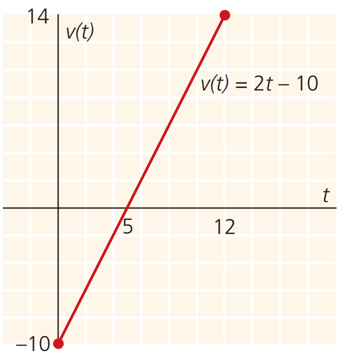
EXAMPLE 4 Discuss the motion of an object moving on a horizontal line if its position is given by $s(t) = t^2 - 10t$, $0 \leq t \leq 12$, where s is in metres and t is in seconds. Include the initial velocity, final velocity, and any acceleration in your discussion.

Solution

The initial position of the object occurs at time $t = 0$. Since $s(0) = 0$, the object starts at the origin.

The velocity at time t is

$$v(t) = s'(t) = 2t - 10$$
$$= 2(t - 5).$$



The object is at rest when $v(t) = 0$.

$$2(t - 5) = 0$$
$$t = 5$$

$v(t) > 0$ for $5 < t \leq 12$, therefore the object is moving to the right.

$v(t) < 0$ if $0 \leq t < 5$, therefore the object is moving to the left.

The initial velocity is $v(0) = -10$.

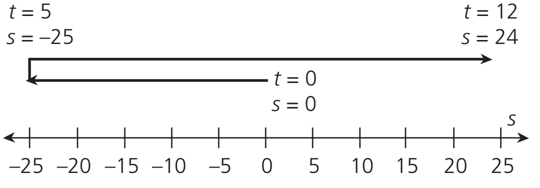
At $t = 12$, $v(12) = 14$.

The acceleration at time t is

$$a(t) = v'(t) = s''(t) = 2.$$

The object moves to the left for $0 \leq t < 5$ and to the right for $5 < t \leq 12$. The initial velocity is -10 m/s, the final velocity is 14 m/s, and the acceleration is 2 m/s^2.

The following diagram is a schematic of the motion. (The actual path of the object is back and forth on a line.)



Motion Under Gravity Near the Surface of the Earth

EXAMPLE 5

A fly ball is hit vertically upward. The position function $s(t)$, in metres, of the ball is $s(t) = -5t^2 + 30t + 1$ where t is in seconds.

a. Find the maximum height reached by the ball.

b. Find the velocity of the ball when it is caught 1 m above the ground.

Solution

a. The maximum height occurs when the velocity of the ball is zero, that is, when the slope of the tangent to the graph is zero.

The velocity function is

$$\begin{aligned} v(t) &= s'(t) \\ &= -10t + 30. \end{aligned}$$

On solving $v(t) = 0$, we obtain $t = 3$. Therefore, the maximum height reached by the ball is $s(3) = 46$ m.

b. When the ball is caught, $s(t) = 1$. To find the time at which this occurs, solve

$$\begin{aligned} 1 &= -5t^2 + 30t + 1 \\ 0 &= -5t(t - 6) \\ t &= 0 \text{ or } t = 6. \end{aligned}$$

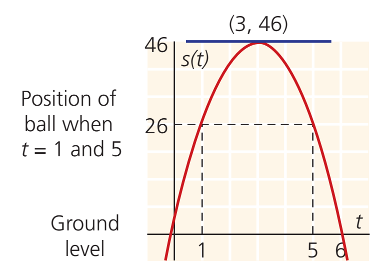
Since $t = 0$ is the time at which the ball left the bat, the time at which the ball was caught is $t = 6$.

The velocity of the ball when it was caught is $v(6) = -30$ m/s.

This negative value is reasonable, since the ball is falling (moving in a negative direction) when it is caught.

Note, however, that the graph of $s(t)$ does not represent the path of the ball. We think of the ball as moving in a straight line along a vertical s-axis, with the direction of motion reversing when $s = 46$.

To see this, note that the ball is at the same height at time $t = 1$, when $s(1) = 26$, and at time $t = 5$, when $s(5) = 26$.



Exercise 5.2

Part A

Communication

1. Explain and discuss the difference in velocity at times $t = 1$ and $t = 5$ for $v(t) = 2t - t^2$.

2. Find the second derivative of each of the following:

a. $y = x^{10} + 3x^6$ b. $f(x) = \sqrt{x}$ c. $y = (1 - x)^2$

Knowledge/Understanding

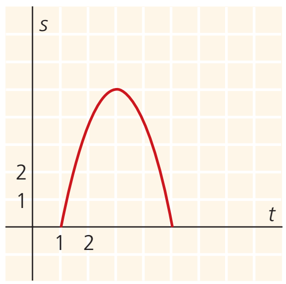
3. For the following position functions, each of which describes the motion of an object along a straight line, find the velocity and acceleration as functions of t, $t \geq 0$.

a. $s(t) = 5t^2 - 3t + 15$ b. $s(t) = 2t^3 + 36t - 10$ c. $s(t) = t - 8 + \frac{6}{t}$

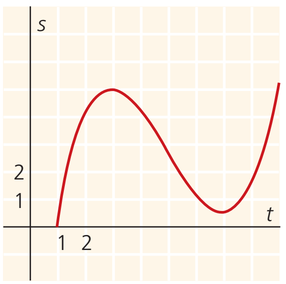
d. $s(t) = (t - 3)^2$ e. $s(t) = \sqrt{t + 1}$ f. $s(t) = \frac{9t}{t + 3}$

4. Consider the following positive time graphs.
   i) When is the velocity zero?
   ii) When is the object moving in a positive direction?
   iii) When is the object moving in a negative direction?

a.



b.



5. A particle moves along a straight line with the equation of motion $s = \frac{1}{3}t^3 - 2t^2 + 3t$, $t \geq 0$. Find the particle's velocity and acceleration at any time, t. When does the direction of the motion of the particle change? When does the particle return to its initial position?

Part B

6. Each function describes the position of an object that moves along a straight line. Determine whether the object is moving in a positive or in a negative direction at time $t = 1$ and at time $t = 4$.

   a. $s(t) = -\frac{1}{3}t^2 + t + 4$ b. $s(t) = t(t - 3)^2$ c. $s(t) = t^3 - 7t^2 + 10t$

7. Starting at $t = 0$, a particle moves along a line so that its position after t seconds is $s(t) = t^2 - 6t + 8$, where s is in metres.
   a. What is its velocity at time t?
   b. When is its velocity zero?

8. When an object is launched vertically from the ground level with an initial velocity of 40 m/s, its position after t seconds will be $s(t) = 40t - 5t^2$ metres above ground level.
   a. When does the object stop rising?
   b. What is its maximum height?

9. An object moves in a straight line, and its position, s, in metres after t seconds is $s(t) = 8 - 7t + t^2$.
   a. Find the velocity when $t = 5$.
   b. Find the acceleration when $t = 5$.

Application **10.** The position function of a moving object is $s(t) = t^{\frac{5}{2}}(7 - t)$, $t \geq 0$, in metres, at time t in seconds.
   a. Find the object's velocity and acceleration at any time, t.
   b. After how many seconds does the object stop?
   c. When does the direction of motion of the object change?
   d. When is its acceleration positive?
   e. When does the object return to its original position?

11. A ball is thrown upwards so that its height, h, in metres above the ground after t seconds is given by $h(t) = -5t^2 + 25t$, $t \geq 0$.
   a. Find the ball's initial velocity.
   b. Find its maximum height.
   c. When does the ball strike the ground and what is its velocity at this time?

12. A dragster races down a 400 m strip in 8 s. Its distance, in metres, from the starting line after t seconds is $s(t) = 6t^2 + 2t$.
   a. Find the dragster's velocity and acceleration as it crosses the finish line.
   b. How fast was it moving 60 m down the strip?

13. For each of the following position functions, discuss the motion of an object moving on a horizontal line where s is in metres and t is in seconds. Make a graph similar to that in Example 4 showing the motion for $t \geq 0$. Find the velocity and acceleration, and determine the extreme positions (farthest left or right) for $t \geq 0$.
   a. $s = 10 + 6t - t^2$
   b. $s = t^3 - 12t - 9$

14. If the position function of an object is $s(t) = t^5 - 10t^2$, at what time, t, in seconds, will the acceleration be zero? Is the object moving towards or away from the origin at that instant?

Thinking/Inquiry/Problem Solving **15.** The distance–time relationship for a moving object is given by $s(t) = kt^2 + (6k^2 - 10k)t + 2k$, where k is a non-zero constant.
   a. Show that the acceleration is constant.
   b. Find the time at which the velocity is zero, and determine the position of the object when this occurs.

Thinking/Inquiry/ Problem Solving

16. Newton's laws of motion imply simple formulas for $s(t)$ and $v(t)$, and the functions of altitude (in metres) and vertical velocity (in m/s): altitude is $s(t) = s_0 + v_0t - 5t^2$, and vertical velocity is $v(t) = v_0 - 10t$, where s_0 is the metres above ground, and v_0 is the upward velocity at $t = 0$. According to the 1998 *Guinness Book of Records*, the roof of the SkyDome in Toronto, Ontario, is the world's only retractable roof. It covers 3.2 ha (8 acres), spans 209 m at its widest point, and rises 86 m. It takes 20 min to retract the roof fully.

Could a major-league pitcher hit the 86 m ceiling of the SkyDome? *Hint:* Assume the pitcher can throw the ball horizontally at 50 m/s, at about 35 m/s straight up, and that he throws the ball from the pitcher's mound.

Part C

17. An elevator is designed to start from a resting position without a jerk. It can do this if the acceleration function is continuous.

a. Show that for the position function $s(t) = \begin{cases} 0, & \text{if } t < 0 \\ \dfrac{t^3}{t^2 + 1}, & \text{if } t \geq 0 \end{cases}$, the acceleration is continuous at $t = 0$.

b. What happens to the velocity and acceleration for very large values of t?

18. An object moves so that its velocity, v, is related to its position, s, according to $v = \sqrt{b^2 + 2gs}$, where b and g are constants. Show that the acceleration of the object is constant.

19. Newton's law of motion for a particle of mass m moving in a straight line says that $F = ma$, where F is the force acting on the particle and a is the acceleration of the particle. In relativistic mechanics, this law is replaced by $F = \dfrac{m_0 \frac{d}{dt} v}{}$, where m_0 is the mass of the particle measured at rest and c is the velocity of light. Show that $F = \dfrac{m_0 a}{\left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{3}{2}}}$.

Section 5.3 — Related Rates

Oil spilled from a tanker spreads in a circle whose area increases at a constant rate of 6 km^2/h. How fast is the radius of the spill increasing when the area is 9π km^2? Knowing the rate of increase of the radius is important in planning the containment operation.

In this section, you will encounter some interesting problems that will help you to understand the applications of derivatives and how they can be used to describe and predict the phenomena of change. In many practical applications, several quantities vary in relation to one another. The rates at which they vary are also related to one another. With calculus, we can describe and calculate such rates.

EXAMPLE 1

When a raindrop falls into a still puddle, ripples spread out in concentric circles from the point where the raindrop hits. The radii of these circles grow at the rate of 3 cm/s.

a. Find the rate of increase of the circumference of one circle.

b. Find the rate of increase of the area of the circle that has an area of 81π cm^2.

Solution

The circumference of a circle is $C = 2\pi r$, and the area of a circle is $A = \pi r^2$.

We are given that $\frac{dr}{dt} = 3$.

a. To find $\frac{dC}{dt}$ at any time, it is necessary to differentiate the equation $C = 2\pi r$ with respect to t.

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}$$

At time t, since $\frac{dr}{dt} = 3,$

$$\frac{dC}{dt} = 2\pi(3)$$

$$= 6\pi.$$

Therefore, the circumference is increasing at a constant rate of 6π cm/s.

b. To find $\frac{dA}{dt}$, differentiate $A = \pi r^2$ with respect to t.

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

We know that $\frac{dr}{dt} = 3$, so we need to determine r.

Since $A = 81\pi$ and $A = \pi r^2$,

then $\pi r^2 = 81\pi$

$r^2 = 81$

$r = 9, r > 0,$

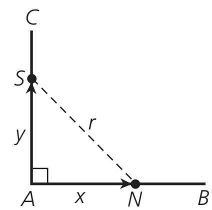
and $\frac{dA}{dt} = 2\pi(9)(3)$

$= 54\pi.$

The area of the circle is increasing at a rate of 54π cm^2/s at the given instant.

EXAMPLE 2 Many related-rate problems involve right triangles and the Pythagorean Theorem. In these problems, the lengths of the sides of the triangle vary with time. These quantities and related rates can be represented quite simply on the Cartesian plane.

Natalie and Shannon start from point A and drive along perpendicular roads AB and AC respectively as shown. Natalie drives at a speed of 45 km/h and Shannon travels at a speed of 40 km/h. If Shannon begins one hour before Natalie, at what rate are their cars separating three hours after Shannon leaves?



Solution

Let x represent the distance Natalie's car has travelled along AB, and let y represent the distance Shannon has travelled along AC.

Therefore $\frac{dx}{dt} = 45$ and $\frac{dy}{dt} = 40$.

Let r represent the distance between the two cars at time t.
Therefore, $x^2 + y^2 = r^2$.
Differentiate both sides of the equation with respect to time.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(r^2)$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

or $$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt} \quad ①$$

Natalie has travelled for 2 h or $2 \times 45 = 90$ km.
Shannon has travelled for 3 h or $3 \times 40 = 120$ km.

The distance between the cars is

$$90^2 + 120^2 = r^2$$
$$r = 150.$$

Substituting into equation ①,

$x = 90, \frac{dx}{dt} = 45, y = 120, \frac{dy}{dt} = 40$, and $r = 150$

$90 \times 45 + 120 \times 40 = 150\frac{dr}{dt}$.

Therefore, the distance between Natalie's and Shannon's cars is increasing at a rate of 59 km/h after two hours.

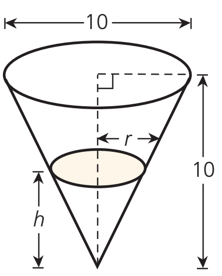
EXAMPLE 3

Water is pouring into an inverted right circular cone at a rate of π m^3/min. The height and the diameter of the base of the cone are both 10 m. How fast is the water level rising when its depth is 8 m?

Solution

Let V denote the volume, r the radius, and h the height of water in the cone at time t. The volume of water in the cone at any time is $V = \frac{1}{3}\pi r^2 h$.

Since we want to find $\frac{dh}{dt}$ when $h = 8$, we solve for r in terms of h from the ratio determined from the similar triangles $\frac{r}{h} = \frac{5}{10}$ or $r = \frac{1}{2}h$.



Substituting into $V = \frac{1}{3}\pi r^2 h$,

we get
$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$
$$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h$$
$$V = \frac{1}{12}\pi h^3.$$

Differentiating with respect to time, we find

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}.$$

At a specific time, when $h = 8$ and $\frac{dV}{dt} = \pi$,

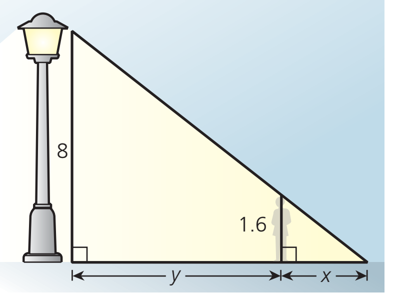
$$\pi = \frac{1}{4}\pi(8)^2\frac{dh}{dt}$$
$$\frac{1}{16} = \frac{dh}{dt}.$$

Therefore, at the moment when the depth of the water is 8 m, the level is rising at 0.0625 m/min.

EXAMPLE 4 A student 1.6 m tall walks directly away from a lamppost at a rate of 1.2 m/s. A light is situated 8 m above the ground on the lamppost. Show that the student's shadow is lengthening at a rate of 0.3 m/s when she is 20 m from the base of the lamppost.

Solution

Let x be the length of her shadow and y be the distance she is from the lamppost, in metres, as shown. Let t denote the time, in seconds.



We are given that $\frac{dy}{dt} = 1.2$ m/s and we wish to determine $\frac{dx}{dt}$ when $y = 20$ m.

To find a relationship between x and y, use similar triangles.

$$\frac{x+y}{8} = \frac{x}{1.6}$$

$$1.6x + 1.6y = 8x$$

$$1.6y = 6.4x$$

Differentiating both sides with respect to t, $1.6\frac{dy}{dt} = 6.4\frac{dx}{dt}$.

When $y = 20$ and $\frac{dy}{dt} = 1.2$,

$$1.6(1.2) = 6.4\frac{dx}{dt}$$

$$\frac{dx}{dt} = 0.3.$$

Therefore, her shadow is lengthening at 0.3 m/s. (Note that the shadow is lengthening at a constant rate, independent of her distance from the lamppost.)

Guidelines for Solving Related-Rate Problems

1. Make a sketch and label the quantities, if applicable.
2. Introduce variables to represent the quantities that change.
3. Identify the quantities to be determined.
4. Find an equation that relates the variables.
5. Implicitly differentiate both sides of the equation with respect to time t, regarding all variables as functions of t.
6. Substitute into the differentiated equation all known values for the variables and their rates of change.
7. Solve the equation for the required rate of change.
8. Write a conclusion that includes the units.

Exercise 5.3

Part A

Communication

1. Express the following statements in symbols:

a. The area, A, of a circle is increasing at a rate of 4 m^2/s.

b. The surface area, S, of a sphere is decreasing at a rate of 3 m^2/min.

c. After travelling for 15 min, the speed of a car is 70 km/h.

d. The x- and y-coordinates of a point are changing at equal rates.

e. The head of a short-distance radar dish is revolving at three revolutions per minute.

Part B

2. The function $T(x) = \frac{200}{1 + x^2}$ represents the temperature in degrees Celsius perceived by a person standing x metres from a fire.

a. If the person moves away from the fire at 2 m/s, how fast is the temperature changing when the person is 5 m away?

b. Using a graphing calculator, determine the distance from the fire when the perceived temperature is changing the fastest.

c. What other calculus techniques could be used to check the result?

Knowledge/Understanding

3. The side of a square is increasing at a rate of 5 cm/s. At what rate is the area changing when the side is 10 cm long? At what rate is the perimeter changing?

4. Each edge of a cube is expanding at a rate of 4 cm/s.

a. How fast is the volume changing when each edge is 5 cm?

b. At what rate is the surface area changing when each edge is 7 cm?

5. One side of a rectangle increases at 2 cm/s, while the other side decreases at 3 cm/s. How fast is the area of the rectangle changing when the first side equals 20 cm and the second side equals 50 cm?

6. The area of a circle is decreasing at the rate of 5 m^2/s when its radius is 3 m.

a. At what rate is the radius decreasing at that moment?

b. At what rate is the diameter decreasing at that moment?

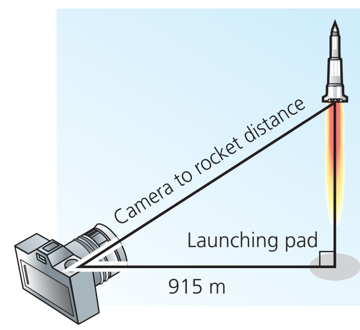
Application

7. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 km^2/h. How fast is the radius of the spill increasing when the area is 9π km^2?

8. The top of a 5 m wheeled ladder rests against a vertical wall. If the bottom of the ladder rolls away from the base of the wall at a rate of $\frac{1}{3}$ m/s, how fast is the top of the ladder sliding down the wall when it is 3 m above the base of the wall?

9. How fast must someone let out line if the kite that she is flying is 30 m high, 40 m away from her horizontally, and continuing to move away from her horizontally at the rate of 10 m/min?

10. If the rocket shown in the figure is rising vertically at 268 m/s when it is 1220 m up, how fast is the camera-to-rocket distance changing at that instant?



11. Two cyclists depart at the same time from a starting point along routes making an angle of $\frac{\pi}{3}$ radians with each other. The first is travelling at 15 km/h, while the second is moving at 20 km/h. How fast are the two cyclists moving apart after 2 h?

12. A spherical balloon is being filled with helium at a rate of 8 cm^3/s. At what rate is its radius increasing
   a. when the radius is 12 cm?
   b. when the volume is 1435 cm^3? (Your answer should be correct to the nearest hundredth.)
   c. when it has been filling for 33.5 s?

13. A cylindrical tank with height 15 m and diameter 2 m is being filled with gasoline at a rate of 500 L/min. At what rate is the fluid level in the tank rising? (1 L = 1000 cm^3). About how long will it take to fill the tank?

Communication

14. If $V = \pi r^2 h$, find $\frac{dV}{dt}$ if r and h are both variables. In your journal, write three problems that involve the rate of change of the volume of a cylinder such that
   i) r is a variable and h is a constant;
   ii) r is a constant and h is a variable;
   iii) r and h are both variables.

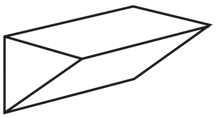
15. The trunk of a tree is approximately cylindrical in shape and has a diameter of 1 m when the height is 15 m. If the radius is increasing at 0.003 m per

annum and the height is increasing at 0.4 m per annum, find the rate of increase of the volume of the trunk.

16. A conical paper cup of radius 5 cm and height 15 cm is leaking water at the rate of 2 cm^3/min. At what rate is the level of water decreasing when the water is 3 cm deep?

17. Derive the formula for the volume of a trough whose cross-section is an equilateral triangle and whose length is 10 m.

18. The cross-section of a water trough is an equilateral triangle with a horizontal top edge. If the trough is 5 m long and 25 cm deep, and water is flowing in at a rate of 0.25 m^3/min, how fast is the water level rising when the water is 10 cm deep at the deepest point?



19. The shadow cast by a man standing 1 m from a lamppost is 1.2 m long. If the man is 1.8 m tall and walks away from the lamppost at a speed of 120 m/min, at what rate is the shadow lengthening after 5 s?

Part C

20. A railroad bridge is 20 m above, and at right angles to, a river. A person in a train travelling at 60 km/h passes over the centre of the bridge at the same instant that a person in a motorboat travelling at 20 km/h passes under the centre of the bridge. How fast are the two people separating 10 s later?

21. Liquid is being poured into the top of a funnel at a steady rate of 200 cm^3/s. The funnel is in the shape of an inverted right circular cone with a radius equal to its height. It has a small hole in the bottom where the liquid is flowing out at a rate of 20 cm^3/s. How fast is the height of the liquid changing when the liquid in the funnel is 15 cm deep?

    At the instant when the height of the liquid is 25 cm, the funnel becomes clogged at the bottom and no more liquid flows out. How fast does the height of the liquid change just after this occurs?

22. A ladder of length l standing on level ground is leaning against a vertical wall. The base of the ladder begins to slide away from the wall. Introduce a coordinate system so that the wall lies along the y-axis, the ground is on the x-axis, and the base of the wall is the origin.

    What is the equation of the path followed by the midpoint of the ladder? What is the equation of the path followed by any point on the ladder? (*Hint:* Let k be the distance from the top of the ladder to the point in question.)

23. A ball is dropped from a height of 20 m, 12 m away from the top of a 20 m lamppost. The ball's shadow, created by the light at the top of the lamppost, is moving along the level ground. How fast is the shadow moving one second after the ball is released?

Section 5.4 — Maximum and Minimum on an Interval

INVESTIGATION

The purpose of this investigation is to determine how the derivative can be used in determining the maximum (largest) value or the minimum (smallest) value of a function on a given interval.

1. For each of the following functions, determine, by completing the square, the value of x that produces a maximum or minimum function value on the given interval.
   a. $f(x) = -x^2 + 6x - 3$, interval $0 \leq x \leq 5$
   b. $f(x) = -x^2 - 2x + 11$, interval $-3 \leq x \leq 4$
   c. $f(x) = 4x^2 - 12x + 7$, interval $-1 \leq x \leq 4$

2. For each function, determine the value of c such that $f'(c) = 0$.

3. Compare the values obtained in Questions 1 and 2 for each function.

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4. Using your calculator, graph each of the following functions and determine all values of x that produce a maximum or minimum function value on the given interval.
   a. $f(x) = x^3 - 3x^2 - 8x + 10$, interval $-2 \leq x \leq 4$
   b. $f(x) = x^3 - 12x + 5$, interval $-3 \leq x \leq 3$
   c. $f(x) = 3x^3 - 15x^2 + 9x + 23$, interval $0 \leq x \leq 4$
   d. $f(x) = -2x^3 + 12x + 7$, interval $-2 \leq x \leq 2$
   e. $f(x) = -x^3 - 2x^2 + 15x + 23$, interval $-4 \leq x \leq 3$

5. For each function in Question 4, determine all values of c such that $f'(c) = 0$.

6. Compare the values obtained in Questions 4 and 5 for each function.

technology

7. From your conclusions in Questions 3 and 6, state a method for using the derivative of a function to determine values of the variable that determine maximum or minimum values of the function.

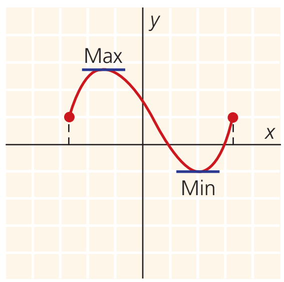
8. Repeat Question 4 for the following functions, using the indicated intervals.
   a. $f(x) = -x^2 + 6x - 3$, interval $4 \leq x \leq 8$
   b. $f(x) = 4x^2 - 12x + 7$, interval $2 \leq x \leq 6$
   c. $f(x) = x^3 - 3x^2 - 9x + 10$, interval $-2 \leq x \leq 6$
   d. $f(x) = x^3 - 12x + 5$, interval $0 \leq x \leq 5$
   e. $f(x) = x^3 - 5x^2 + 3x + 7$, interval $-2 \leq x \leq 5$

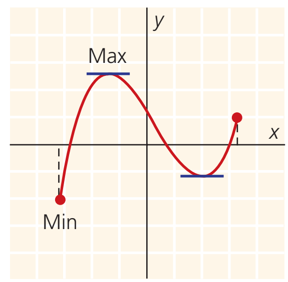
9. In Questions 3 and 6, you saw that a maximum or minimum can occur at points $(c, f(c))$ where $f'(c) = 0$. From your observations in Question 8, state other values of the variable that can produce a maximum or minimum in a given interval.

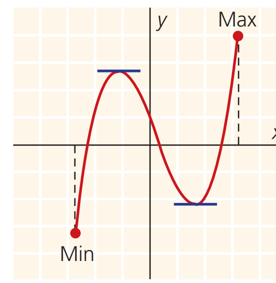
Checkpoint: Check Your Understanding

The maximum value of a function that has a derivative at all points in an interval occurs at a "peak" $(f'(c) = 0)$ or at an end point of the interval. The minimum value occurs at a "valley" $(f'(c) = 0)$ or at an end point. This is true no matter how many peaks and valleys the graph has in the interval.

In the following three graphs, the derivative equals zero at two points.







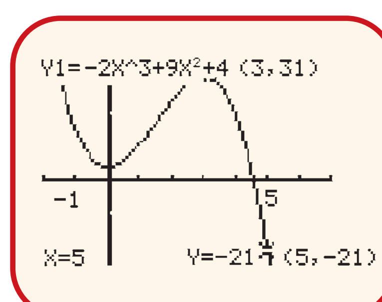
Algorithm for Maximum or Minimum (Extreme Values)

If a function $f(x)$ has a derivative at every point in the interval $a \le x \le b$, calculate $f(x)$ at

- **all points in the interval $a \le x \le b$ where $f'(x) = 0$;**
- **the end points $x = a$ and $x = b$.**

The maximum value of $f(x)$ on the interval $a \le x \le b$ is the largest of these values, and the minimum value of $f(x)$ on the interval is the smallest of these values.

EXAMPLE 1 Find the extreme values of the function $f(x) = -2x^3 + 9x^2 + 4$ on the interval $-1 \le x \le 5$.



Solution

The derivative is $f'(x) = -6x^2 + 18x$.

If we set $f'(x) = 0$, we obtain

$$-6x(x - 3) = 0,$$

so $x = 0$ or $x = 3$.

Both values lie in the given domain.

We can then evaluate $f(x)$ for these values and at the end points $x = -1$ and $x = 5$, to obtain

$$\begin{aligned} f(-1) &= 15 \\ f(0) &= 4 \\ f(3) &= 31 \\ f(5) &= -21. \end{aligned}$$

Therefore, the maximum value of $f(x)$ on the interval $-1 \leq x \leq 5$ is $f(3) = 31$, and the minimum value is $f(5) = -21$.

EXAMPLE 2

The amount of current in an electrical system is given by the function $C(t) = -t^3 + t^2 + 21t$, where t is the time in seconds and $0 \leq t \leq 5$. Determine the times at which the current is maximal and minimal and the amount of current in the system at these times.

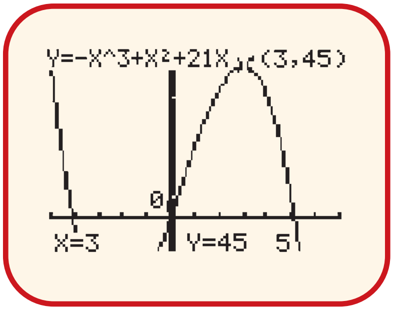
Solution

The derivative is $\frac{dC}{dt} = -3t^2 + 2t + 21$.

If we set $\frac{dC}{dt} = 0$, we obtain

$$\begin{aligned} 3t^2 - 2t - 21 &= 0 \\ (3t + 7)(t - 3) &= 0, \end{aligned}$$

therefore, $t = -\frac{7}{3}$ or 3.



Only $t = 3$ is in the given interval, so we evaluate $C(t)$ at $t = 0$, $t = 3$, and $t = 5$ as follows:

$$\begin{aligned} C(0) &= 0 \\ C(3) &= -3^3 + 3^2 + 21(3) = 45 \\ C(5) &= -5^3 + 5^2 + 21(5) = 5. \end{aligned}$$

The maximum is 45 units at time $t = 3$ s, and the minimum is zero units at time $t = 0$ s.

EXAMPLE 3

The amount of light intensity on a point is given by the function $I(t) = \frac{t^2 + 2t + 16}{t + 2}$, where t is the time in seconds and $0 \leq t \leq 14$. Determine the time of minimal intensity.

Solution

Note that the function is not defined for $t = -2$. Since this value is not in the given interval, we need not worry about it.
The derivative is

$$I(t) = \frac{(2t + 2)(t + 2) - (t^2 + 2t + 16)(1)}{}$$

$$= \frac{2t^2 + 6t + 4 - t^2 - 2t - 16}{}$$

$$= \frac{t^2 + 4t - 12}{}.$$

If we set $I'(t) = 0$, we obtain

$$t^2 + 4t - 12 = 0$$
$$(t + 6)(t - 2) = 0$$
$$t = -6 \text{ or } t = 2.$$

Only $t = 2$ is in the given interval, so we evaluate $I(t)$ for $t = 0$, 2, and 14:

$$I(0) = 8$$

$$I(2) = \frac{4 + 4 + 16}{4} = 6$$

$$I(14) = \frac{14^2 + 2(14) + 16}{16} = 15.$$

Note that the calculation can be greatly reduced by rewriting the function, as shown:

$$I(t) = \frac{t^2 + 2t}{t + 2} + \frac{16}{t + 2}$$

$$= t + 16(t + 2)^{-1}.$$

Then $\quad I'(t) = 1 - 16(t + 2)^{-2}$

$$= 1 - \frac{16}{(t + 2)^2}.$$

Setting $\quad I'(t) = 0$ gives

$$1 = \frac{16}{(t + 2)^2}$$

$$t^2 + 4t + 4 = 16$$
$$t^2 + 4t - 12 = 0.$$

As before, $t = -6$ or $t = 2$.

The evaluations are also simplified:

$$I(0) = 0 + 8 = 8$$

$$I(2) = 2 + \frac{16}{4} = 6$$

$$I(14) = 14 + \frac{16}{16} = 15.$$

Exercise 5.4

Part A

Communication

1. State, with reasons, why the maximum/minimum algorithm can or cannot be used to determine the maximum and minimum values for the following:

a. $y = x^3 - 5x^2 + 10$ on $-5 \leq x \leq 5$

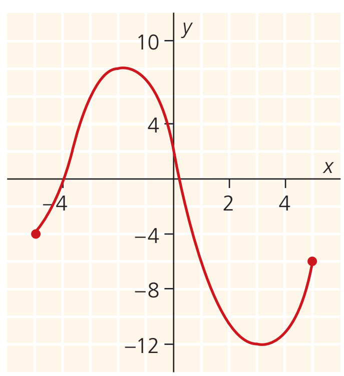
b. $y = \frac{3x}{x - 2}$ on $-1 \leq x \leq 3$

c. $y = \frac{x}{x^2 - 4}$ on $0 \leq x \leq 5$

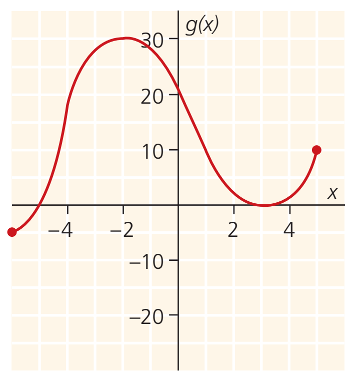
d. $y = \frac{x^2 - 1}{x + 3}$ on interval $-2 \leq x \leq 3$

2. State the value of the maximum and the minimum for each function. In each of the following graphs, the function is defined in the interval shown.

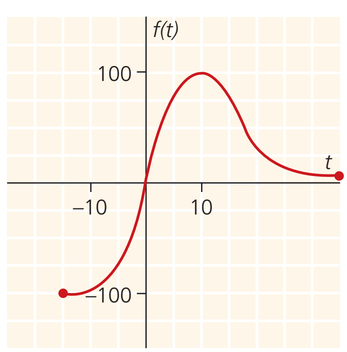
a.



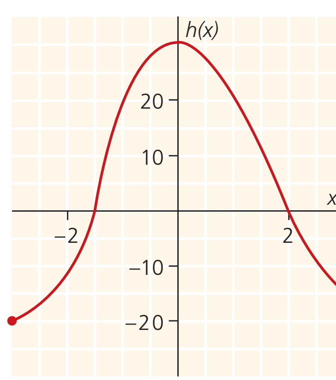
b.



c.



d.



Knowledge/Understanding

3. Find the maximum or minimum value of each function on the given interval, using the algorithm for maximum or minimum values. Illustrate your results by sketching the graph of each function.

technology

a. $f(x) = x^2 - 4x + 3, 0 \leq x \leq 3$

b. $f(x) = (x - 2)^2, 0 \leq x \leq 2$

c. $f(x) = x^3 - 3x^2, -1 \leq x \leq 3$

d. $f(x) = x^3 - 3x^2, -2 \leq x \leq 1$

e. $f(x) = 2x^3 - 3x^2 - 12x + 1, -2 \leq x \leq 0$

f. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x, 0 \leq x \leq 4$

Part B

4. Find the extreme values of each function on the given interval, using the algorithm for maximum or minimum values.

a. $f(x) = x + \frac{4}{x}, 1 \leq x \leq 10$

b. $f(x) = 4\sqrt{x} - x, 2 \leq x \leq 9$

c. $f(x) = \frac{1}{x^2 - 2x + 2}, 0 \leq x \leq 2$

d. $f(x) = 3x^4 - 4x^3 - 36x^2 + 20, -3 \leq x \leq 4$

e. $f(x) = \frac{4x}{x^2 + 1}, -2 \leq x \leq 4$

f. $f(x) = \frac{4x}{x^2 + 1}, 2 \leq x \leq 4$

5. a. An object moves in a straight line. Its velocity in m/s at time t is $v(t) = \frac{4t^2}{4 + t^3}, t \geq 0$. Find the maximum and minimum velocities over the time interval $1 \leq t \leq 4$.

b. Repeat part **a** if $v(t) = \frac{4t^2}{1 + t^2}, t \geq 0$.

Application

6. A swimming pool is treated periodically to control the growth of bacteria. Suppose that t days after a treatment, the concentration of bacteria per cubic centimetre is $C(t) = 30t^2 - 240t + 500$. Find the lowest concentration of bacteria during the first week after the treatment.

7. The fuel efficiency, E, (in litres per hundred kilometres) of a car driven at speed v (in km/h) is $E(v) = \frac{1600v}{v^2 + 6400}$.

a. If the speed limit is 100 km/h, find the legal speed that will maximize the fuel efficiency.

b. Repeat part **a** using a speed limit of 50 km/h.

8. The concentration $C(t)$ (in milligrams per cubic centimetre) of a certain medicine in a patient's bloodstream is given by $C(t) = \frac{0.1t}{(t+3)^2}$, where t is the number of hours after the medicine is taken. Find the maximum and minimum concentrations between the first and sixth hours after the patient is given the medicine.

9. Technicians working for the Ministry of Natural Resources have found that the amount of a pollutant in a certain river can be represented by $P(t) = 2t + \frac{1}{(162t+1)}$, $0 \leq t \leq 1$, where t is the time (in years) since a clean-up campaign started. At what time was the pollution at its lowest level?

10. A truck travelling at x km/h, where $30 \leq x \leq 120$, uses gasoline at the rate of $r(x)$ L/km, where $r(x) = \frac{1}{400}\left(\frac{4900}{x} + x\right)$. If fuel costs \$0.45/L, what speed will result in the lowest fuel cost for a trip of 200 km? What is the lowest total cost for the trip?

Part C

Thinking/Inquiry/ Problem Solving

11. In a certain manufacturing process, when the level of production is x units, the cost of production (in dollars) is $C(x) = 3000 + 9x + 0.05x^2$, $1 \leq x \leq 300$.

What level of production x will minimize the unit cost $U(x) = \frac{C(x)}{x}$? Keep in mind that the production level must be an integer.

12. Repeat Question 11 using a cost of production of $C(x) = 6000 + 9x + 0.05x^2$, $1 \leq x \leq 300$.

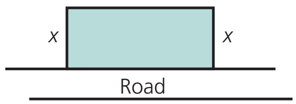
Section 5.5 — Optimization Problems

We frequently encounter situations in which we are asked to do the best we can. Such a request is vague unless we are given some conditions. Asking us to minimize the cost of making tables and chairs is not clear. Asking us to make the maximum number of tables and chairs possible so that the costs of production are minimized and given that the amount of material available is restricted allows us to construct a function describing the situation. We can then determine the minimum (or maximum) of the function.

Such a procedure is called **optimization.** To optimize a situation is to realize the best possible outcome, subject to a set of restrictions. Because of these restrictions, the domain of the function is usually restricted. As you have seen earlier, in such situations, the absolute maximum or minimum can be identified through the use of calculus, but might also occur at the ends of the domain.

A farmer has 800 m of fencing and wishes to enclose a rectangular field. One side of the field is against a country road which is already fenced, so the farmer needs to fence only the remaining three sides of the rectangular field. The farmer wishes to enclose the maximum possible area and wishes to use all of the fencing. How does he determine the dimensions that achieve this?

The farmer can achieve his goal by determining a function that describes the area, subject to the condition that the amount of fencing used is to be exactly 800 m, and by finding the absolute maximum of the function. To do so, he would proceed as follows:



Let the width of the enclosed area be x m. Then the length of the area is $(800 - 2x)$ m. The area of the field can be represented by the function $A(x)$ where

$$\begin{aligned} A(x) &= x(800 - 2x) \\ &= 800x - 2x^2. \end{aligned}$$

The domain of the function is $0 \leq x \leq 400$ since the amount of fencing is 800 m. To find the minimum and maximum values, determine $A'(x)$:

$$A'(x) = 800 - 4x.$$

Setting $A'(x) = 0$, we obtain $800 - 4x = 0$

$$x = 200.$$

The minimum and maximum values can occur at $x = 200$ or at the ends of the domain, $x = 0$ and $x = 400$.

$$\begin{aligned} A(0) &= 0 \\ A(200) &= 200(800 - 400) \\ &= 80\ 000 \\ A(400) &= 400(800 - 800) \\ &= 0 \end{aligned}$$

The maximum area he can enclose is 80 000 m^2, within a field 200 m by 400 m. The procedure used here can be summarized as follows:

An Algorithm for Solving Optimization Problems

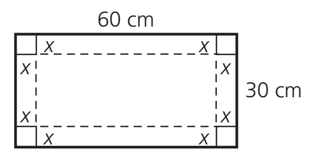
1. Understand the problem and identify quantities that can vary. Determine a function that represents the quantity to be optimized. Be sure that this function only depends on one variable.
2. Whenever possible, draw a diagram, labelling the given and required quantities.
3. Determine the domain of the function to be optimized, using information given in the problem.
4. Use the algorithm for extreme values to find the absolute maximum or minimum function value on the domain.
5. Use the results of step 4 to answer the original problem.

EXAMPLE 1

A piece of sheet metal 60 cm by 30 cm is to be used to make a rectangular box with an open top. Find the dimensions that will give the box with the largest volume.

Solution

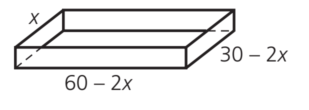
From the diagram, making the box requires that the four corner squares be cut out and discarded. Let each side of the squares be x cm.



Therefore, height $= x$

$$\begin{aligned} \text{length} &= 60 - 2x \\ \text{width} &= 30 - 2x. \end{aligned}$$

Since all dimensions are non-negative, $0 \leq x \leq 15$.



The volume of the box is given by the function $V(x)$, where

$$\begin{aligned} V(x) &= x(60 - 2x)(30 - 2x) \\ &= 4x^3 - 180x^2 + 1800x \end{aligned}$$

For extreme values, set $V'(x) = 0$.

$$\begin{aligned} V'(x) &= 12x^2 - 360x + 1800 \\ &= 12(x^2 - 30x + 150) \end{aligned}$$

Setting $V'(x) = 0$, we obtain $x^2 - 30x + 150 = 0$.

$$\begin{aligned} x &= \frac{30 \pm \sqrt{300}}{2} \\ &= 15 \pm 5\sqrt{3} \\ x &\doteq 23.7 \text{ or } x \doteq 6.3 \end{aligned}$$

Since $x \leq 15$, $x = 15 - 5\sqrt{3} \doteq 6.3$.

To find the largest volume, substitute $x = 0$, 6.3, and 15 in $V(x) = 4x^3 - 180x^2 + 1800x$.

$$\begin{aligned} V(0) &= 0 \\ V(6.3) &= 4(6.3)^3 - 180(6.3)^2 + 1800(6.3) \\ &\doteq 5196 \\ V(15) &= 0 \end{aligned}$$

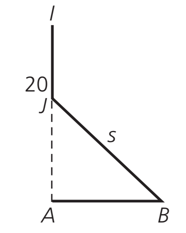
The maximum volume is obtained by cutting out corner squares of side 6.3 cm. The length of the box is $60 - 2 \times 6.3 = 47.4$ cm, the width is $30 - 2 \times 6.3 = 17.4$ cm, and the height is 6.3 cm.

EXAMPLE 2

Ian and Ada are both training for a marathon. Ian's house is located 20 km north of Ada's house. At 9:00 one Saturday morning, Ian leaves his house and jogs south at 8 km/h. At the same time, Ada leaves her house and jogs east at 6 km/h. When are Ian and Ada closest together, given that they both run for 2.5 h?

Solution

If Ian starts at point I, he reaches point J after time t hours. Then $IJ = 8t$ km and $JA = (20 - 8t)$ km.



If Ada starts at point A, she reaches point B after t hours and $AB = 6t$ km.

Now the distance they are apart is $s = JB$, and s can be expressed as a function of t by

$$\begin{aligned} s(t) &= \sqrt{JA^2 + AB^2} \\ &= \sqrt{(20 - 8t)^2 + (6t)^2} \\ &= \sqrt{100t^2 - 320t + 400}. \end{aligned}$$

The domain for t is $0 \leq t \leq 2.5$.

$$\begin{aligned} s'(t) &= \frac{1}{2}(100t^2 - 320t + 400)^{-\frac{1}{2}}(200t - 320) \\ &= \frac{100t - 160}{} \end{aligned}$$

To obtain a minimum or maximum value, let $s'(t) = 0$.

$$\frac{100t - 160}{} = 0$$

$$\begin{aligned} 100t - 160 &= 0 \\ t &= 1.6 \end{aligned}$$

Using the algorithm for extreme values,

$$\begin{aligned} s(0) &= \sqrt{400} \\ &= 20 \\ s(1.6) &= \sqrt{100(1.6)^2 - 320(1.6) + 400} \\ &= \sqrt{144} \\ &= 12 \\ s(2.5) &= \sqrt{225} \\ &= 15. \end{aligned}$$

Therefore, the minimum value of $s(t)$ is 12 km and occurs at time 10:36.

Exercise 5.5

Part A

1. A piece of wire 100 cm long is to be bent to form a rectangle. Determine the rectangle of maximum area.

Communication

2. Discuss the result of maximizing the area of a rectangle given a fixed perimeter.

Knowledge/Understanding

3. A farmer has 600 m of fence and he wants to enclose a rectangular field beside the river on his property. Find the dimensions of the field so that a maximum area is enclosed. (Fencing is required only on three sides.)

Application

4. A rectangular piece of cardboard 100 cm by 40 cm is to be used to make a rectangular box with an open top. Find the dimensions (to one decimal place) for the box with the largest volume.

Part **B**

5. The volume of a square-based rectangular cardboard box is to be 1000 cm^3. Find the dimensions so that the quantity of material used to manufacture all 6 faces is a minimum. Assume that there will be no waste material. The machinery available cannot fabricate material smaller in length than 2 cm.

6. Find the area of the largest rectangle that can be inscribed inside a semicircle with radius of 10 units. Place the length of the rectangle along the diameter.

Application

7. A cylindrical-shaped tin can is to have a capacity of 1000 cm^3.

a. Find the dimensions of the can that require the minimum amount of tin. (Assume no waste material.) The marketing department has specified that the smallest can the market will accept has a diameter of 6 cm and a height of 4 cm.

b. Express the answer for part **a** as a ratio of height to diameter.

Thinking/Inquiry/ Problem Solving

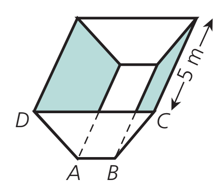
8. a. Find the area of the largest rectangle that can be inscribed in a right triangle with legs adjacent to the right angle of lengths 5 cm and 12 cm. The two sides of the rectangle lie along the legs.

b. Repeat part **a** with the right triangle that has sides 8 cm by 15 cm.

c. Hypothesize a conclusion for any right triangle.

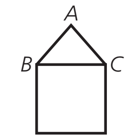
Thinking/Inquiry/ Problem Solving

9. a. An isosceles trapezoidal drainage gutter is to be made so that the angles at A and B, in the cross-section $ABCD$ are each 120º. If the 5-m-long sheet of metal that has to be bent to form the open topped gutter has a width of 60 cm, then find the dimensions so that the cross-sectional area will be a maximum.



b. Calculate the maximum volume of water that can be held by this gutter.

10. A piece of window framing material is 6 m long. A carpenter wants to build a frame for a rural gothic style window where ΔABC is equilateral. The window must fit inside a space 1 m wide and 3 m high.



a. Find the dimensions that should be used for the 6 pieces so that the
maximum amount of light will be admitted. Assume no waste of material for corner cuts, etc.

b. Would the carpenter get more light if he built a window in the shape of an equilateral triangle only? Explain.

11. A train leaves the station at 10:00 and travels due south at a speed of 60 km/h. Another train has been heading due west at 45 km/h and reaches the same station at 11:00. At what time were the two trains closest together?

Part C

Thinking/Inquiry/
Problem Solving

12. In Question 8, you looked at two specific right triangles and observed that the rectangle of maximum area that could be inscribed inside the triangle always had dimensions equal to half the lengths of the sides adjacent to the rectangle. Prove that this is true for any right triangle.

13. Prove that any cylindrical can of volume k cubic units that is to be made using a minimum amount of material must have the height equal to the diameter.

14. A piece of wire 100 cm long is cut into two pieces. One piece is bent to form a square, and the other is bent to form a circle. How should the wire be cut so that the total area enclosed is

    a. a maximum?

    b. a minimum?

15. Determine the minimal distance from point $(-3, 3)$ to the curve given by $y = (x - 3)^2$.

16. A chord joins any two points A and B on the parabola whose equation is $y^2 = 4x$. If C is the midpoint of AB, and CD is drawn parallel to the x-axis to meet the parabola at D, prove that the tangent at D is parallel to chord AB.

17. A rectangle lies in the first quadrant with one vertex at the origin and two of the sides along the coordinate axes. If the fourth vertex lies on the line defined by $x + 2y - 10 = 0$, then find the maximum area of the rectangle.

18. The base of a rectangle lies along the x-axis, and the upper two vertices are on the curve defined by $y = k^2 - x^2$. Find the maximum area of the rectangle.

Section 5.6 — Optimizing in Economics and Science

In the world of business, it is extremely important to manage costs effectively. Good control will allow for a minimization of costs and a maximization of profit. At the same time, there are human considerations. If your company is able to maximize profit but antagonizes its customers or employees in the process, there may be a very significant penalty to pay in the future. For this reason, it may be important that, in addition to any mathematical constraints, you consider other, more practical constraints on the domain when you are constructing a workable function.

The following examples will illustrate economic situations and the domain constraints that you may encounter.

EXAMPLE 1

A cylindrical chemical storage tank with a capacity of 1000 m^3 is to be constructed in a warehouse that is 12 m by 15 m and has a height of 11 m. The specifications call for the base to be made of sheet steel, which costs \$100/m^2, the top of sheet steel, which costs \$50/m^2, and the wall of sheet steel costing \$80/m^2.

a. Determine whether it is possible for a tank of this capacity to fit in the warehouse. If it *is* possible, find the restrictions on the radius.

b. Determine, if the tank is possible, the proportions of the tank that meet the conditions and that minimize the cost of construction.

All calculations should be accurate to two decimal places.

Solution

a. The radius of the tank cannot exceed 6 m, and the maximum height is 11 m. The volume, using $r = 6$ and $h = 11$, is

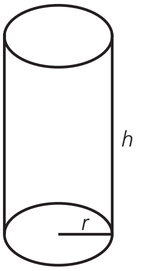
$$\begin{aligned} V &= \pi r^2 h \\ &\cong 1244. \end{aligned}$$

It is possible to build a tank of 1000 m^3.
There are limits on the radius and the height.
Clearly, $r \leq 6$.
Also, if $h = 11$, then $\pi r^2(11) \geq 1000$

$$r \geq 5.38.$$

The tank can be constructed to fit in the warehouse. Its radius must be $5.38 \leq r \leq 6$.

b. If the height is h m and the radius is r m, then
- the cost of the base is $\$100(\pi r^2)$,
- the cost of the top is $\$50(\pi r^2)$, and
- the cost of the wall is $\$80(2\pi rh)$.



The cost of the tank is $C = 150\pi r^2 + 160\pi rh$.
Here we have two variable quantities, r and h.

However, since $V = \pi r^2 h = 1000$,

$$h = \frac{1000}{\pi r^2}.$$

Substituting for h, we have a cost function in terms of r:

$$C(r) = 150\pi r^2 + 160\pi r\left(\frac{1000}{\pi r^2}\right)$$

or $$C(r) = 150\pi r^2 + \frac{160\,000}{r}.$$

From part **a,** we know that the domain is $5.38 \leq r \leq 6$.
For critical points, set $C'(r) = 0$.

$$300\pi r - \frac{160\,000}{r^2} = 0$$

$$300\pi r = \frac{160\,000}{r^2}$$

$$r^3 = \frac{1600}{3\pi}$$

$$r \cong 5.54$$

This value is within the given domain, so we use the algorithm for maximum and minimum.

$$C(5.38) = 150\pi(5.38)^2 + \frac{160\,000}{5.38} \cong 43\,380$$

$$C(5.54) = 150\pi(5.54)^2 + \frac{160\,000}{5.54} \cong 43\,344$$

$$C(6) = 150\pi(6^2) + \frac{160\,000}{6} \cong 43\,631$$

The minimal cost is \$43 344 with a tank of radius 5.54 m and a height of $\frac{1000}{\pi(5.54)^2} = 10.37$ m.

EXAMPLE 2

A commuter train carries 2000 passengers daily from a suburb into a large city. The cost to ride the train is \$7.00 per person. Market research shows that 40 fewer people would ride the train for each \$0.10 increase in the fare, and 40 more people would ride the train for each \$0.10 decrease. If the capacity of the train is 2600 passengers, and contracts with the rail employees require that at least 1600 passengers be carried, what fare should the railway charge to get the largest possible revenue?

Solution

In order to maximize revenue, we require a revenue function. We know that

$$\text{revenue} = (\text{number of passengers}) \times (\text{fare per passenger}).$$

In forming a revenue function, the most straightforward choice for the independent variable comes from noticing that both the number of passengers and the fare per passenger change with each \$0.10 increase or decrease in the fare. If we let x represent the number of \$0.10 increases in the fare (e.g., $x = 3$ represents a \$0.30 increase in the fare, while $x = -1$ represents a \$0.10 decrease in the fare), then we can write expressions for both the number of passengers and the fare per passenger in terms of x, as follows:

$$\text{the fare per passenger is } 7 + 0.10x$$
$$\text{the number of passengers is } 2000 - 40x.$$

Since the number of passengers must be at least 1600, $2000 - 40x \geq 1600$, and $x \leq 10$, and since the number of passengers cannot exceed 2600,
$2000 - 40x \leq 2600$, and $x \geq -15$.
The domain is $-15 \leq x \leq 10$.

The revenue function is

$$\begin{aligned} R(x) &= (7 + 0.10x)(2000 - 40x) \\ &= -4x^2 - 80x + 140\,000. \end{aligned}$$

From a practical standpoint, we also require that x be an integer, in order that the fare only varies by increments of \$0.10. We do not wish to consider fares that are other than multiples of 10 cents.

Therefore the problem is now to find the absolute maximum value of the revenue function.

$$\begin{aligned} R(x) &= (7 + 0.10x)(2000 - 40x) \\ &= -4x^2 - 80x + 14\,000 \end{aligned}$$

on the interval $-15 \leq x \leq 10$, where x must be an integer.

$$\begin{aligned} R'(x) &= -8x - 80 \\ R'(x) &= 0, \text{ when } -8x - 80 = 0 \\ x &= -10 \end{aligned}$$

$R'(x)$ is never undefined. The only critical point for R occurs at $x = 10$, which is in the domain. To determine the absolute maximum revenue, we evaluate

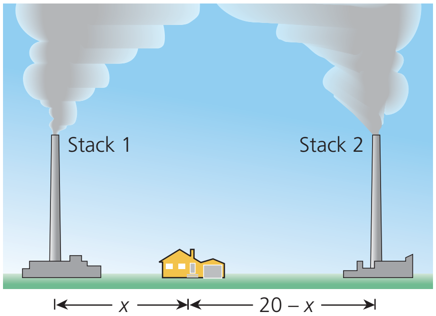
$$\begin{aligned} R(-15) &= -4(-15)^2 - 80(-15) + 14\,000 \\ &= 11\,900 \\ R(-10) &= -4(-10)^2 - 80(-10) + 14\,000 \\ &= 14\,400 \\ R(10) &= -4(10)^2 - 80(10) + 14\,000 \\ &= 12\,800. \end{aligned}$$

Therefore, the maximum revenue occurs when there are -10 fare increases of \$0.10 each, or a fare decrease of $10(0.10) = \$1.00$. At a fare of \$6.00, the daily revenue is \$14 400 and the number of passengers is $2000 - 40(-10) = 2400$.

EXAMPLE 3

In this example, we consider soot deposits from a smokestack. Suppose that a smokestack deposits soot on the ground with a concentration that is inversely proportional to the square of the distance from the foot of the stack (we ignore the height of the stack). For an object located x km from a smokestack, the concentration of soot is modelled by the function $S(x) = \frac{k}{x^2}$, where $k > 0$ is the constant of proportionality that depends on the quantity of smoke emitted by the stack.

If two smokestacks are located 20 km apart, and one is emitting 7 times as much smoke as the other, where on the (straight line) road between the two stacks should a building be built so that the concentration of soot deposit is minimal?



Solution

Let the distance from the building to the stack with lesser emissions be x km. Then the distance from the building to the second stack is $(20 - x)$ km.

Let the constant of proportionality of the first stack be $k > 0$.
Then the constant of proportionality of the second stack (with greater emissions) is 7 k.

The concentration of soot deposit from the first stack is $\frac{k}{x^2}$.

The concentration of soot deposit from the second stack is $\frac{7k}{(20 - x)^2}$.

The total concentration is given by

$$S(x) = \frac{k}{x^2} + \frac{7k}{(20 - x)^2},$$

and since the building is between the stacks, $0 < x < 20$.

To find the critical points of $S(x)$, set $S'(x) = 0$.

$$S'(x) = -\frac{2k}{x^3} + \frac{14k}{(20 - x)^3}$$

If $S'(x) = 0$, $-\frac{2k}{x^3} + \frac{14k}{(20 - x)^3} = 0$

or $$\frac{7k}{(20 - x)^3} = \frac{k}{x^3}$$ (and k has no impact)

$$
\begin{aligned}
7x^3 &= (20 - x)^3 \\
\sqrt[3]{7}x &= 20 - x \\
(1 + \sqrt[3]{7})x &= 20 \\
x &= \frac{20}{\qquad} \\
&\cong 6.9.
\end{aligned}
$$

There is one critical value at $x \cong 6.9$.
Since there are no fixed domain end points, we use the first derivative test to determine whether this gives a minimal value.

If $0 < x < 6.9$,
$$
\begin{aligned}
S'(x) &= -\frac{2k}{x^3} + \frac{14k}{(20 - x)^3} \\
&= -2k\left[\frac{1}{x^3} - \frac{7}{(20 - x)^3}\right] \\
&< 0.
\end{aligned}
$$

Then $S(x)$ is decreasing.

If $6.9 < x < 20$,
$$
\begin{aligned}
S'(x) &= -2k\left[\frac{1}{x^3} - \frac{7}{(20 - x)^3}\right] \\
&> 0.
\end{aligned}
$$

Then $S(x)$ is increasing.
Then a relative minimum occurs at $x = 6.9$, and by the test it is an absolute minimum. The building should be 6.9 km from the smokestack with fewer emissions.

In summary, when solving real-life optimization problems, there are often many factors that can affect the required functions and their domains. Such factors may not be obvious from the statement of the problem. We must do research and ask many questions to address all of the factors. Solving an entire problem is a series of many steps, and optimization using calculus techniques is only one step that is used in determining a solution.

Exercise 5.6

Part A

Knowledge/
Understanding

1. The cost, in dollars, to produce x litres of maple syrup for the Elmira Maple Syrup Festival is $C(x) = 75(\sqrt{x} - 10)$, $x \geq 400$.
   a. What is the average cost of producing 625 L?
   b. The marginal cost is $C'(x)$, and similarly, the marginal revenue is $R'(x)$. What is the marginal cost at 1225 L?
   c. How much production is needed to achieve a marginal cost of \$0.50/L?

Application 2. A sociologist determines that a foreign-language student has learned $N(t) = 20t - t^2$ vocabulary terms after t hours of uninterrupted study.

a. How many terms are learned between times $t = 2$ and $t = 3$ h?

b. What is the rate in terms per hour at which the student is learning at time $t = 2$ h?

c. What is the maximum rate in terms per hour at which the student is learning?

3. A researcher found that the level of antacid in a person's stomach t minutes after a certain brand of antacid tablet is taken is found to be $L(t) = \frac{6t}{t^2 + 2t + 1}$.

a. Find the value of t for which $L'(t) = 0$.

b. Find $L(t)$ for the value found in part **a**.

c. Using your graphing calculator, graph $L(t)$.

d. From the graph, what can you predict about the level of antacid in a person's stomach after 1 min?

e. What is happening to the level of antacid in a person's stomach from $2 \leq t \leq 8$ min?

4. The running cost, C, in dollars per hour for an airplane cruising at a height of h metres and an air speed of 200 km/h is given by $C = 4000 + \frac{h}{15} + \frac{15\,000\,000}{h}$ for the domain $1000 \leq h \leq 20\,000$. Find the height at which the operating cost is at a minimum and find the operating cost per hour.

Application 5. A rectangular piece of land is to be fenced in using two kinds of fencing. Two opposite sides will be fenced using standard fencing that costs \$6/m, while the other two sides will require heavy-duty fencing that costs \$9/m. What are the dimensions of the rectangular lot of greatest area that can be fenced in for a cost of \$9000?

Thinking/Inquiry/ Problem Solving 6. A 20 000 m^3 rectangular cistern is to be made from reinforced concrete so that the interior length will be twice the height. If the cost is \$40/m^2 for the base, \$100/m^2 for the side walls, and \$200/m^2 for the roof, then find the interior dimensions (correct to one decimal place) that will keep the cost to a minimum. To protect the water table, the building code specifies that no excavation can be more than 22 m deep. It also specifies that all cisterns must be at least 1 m in depth.

7. The cost of producing an ordinary cylindrical tin can is determined by the materials used for the wall and the end pieces. If the end pieces are twice as expensive per square centimetre as the wall, find the dimensions (to the nearest millimetre) to make a 1000 cm^3 can at minimal cost.

8. A lighthouse, L, is located on a small island 4 km west of point A on a straight north-south coastline. A power cable is to be laid from L to the nearest source of power at point B on the shoreline, 12 km north of point A. The cost of laying cable under water is \$6000/km and the cost of laying cable along the shoreline is \$2000/km. To minimize the cost, the power line will be built from L underwater to a point C on the shoreline and then along the shoreline from C to B. Find the location of point C (to the nearest metre) on the shoreline where the power cable should enter the water.

9. A bus service carries 10 000 people daily between Cyberville and Steeptown, and the company has space to serve up to 15 000 people per day. The cost to ride the bus is \$20. Market research shows that if the fare is increased by \$0.50, 200 fewer people will ride the bus. What fare should be charged to get maximum revenue, given that the bus company must have at least \$130 000 in fares a day to cover operating costs.

10. The fuel cost per hour for running a ship is approximately one half the cube of the speed plus additional fixed costs of \$216 per hour. Find the most economical speed to run the ship for a 500-nautical-mile trip. *Note:* This assumes that there are no major disturbances such as heavy tides or stormy seas.

11. A truck crossing the prairies at a constant speed of 110 km/h gets 8 km/L of gas. Gas costs \$0.68/L. The truck loses 0.10 km/L in fuel efficiency for each km/h increase in speed. Drivers are paid \$35/h in wages and benefits. Fixed costs for running the truck are \$15.50/h. If a trip of 450 km is planned, what speed will minimize operating expenses?

Communication

12. Your neighbours operate a successful bake shop. One of their specialties is a very rich whipped-cream-covered cake. They buy the cakes from a supplier who charges \$6.00 per cake, and they sell 200 cakes weekly at a price of \$10.00 each. Research shows that profit from the cake sales can be increased by increasing the price. Unfortunately, for every increase of \$0.50 cents, sales will drop by seven.

a. What is the optimal sales price for the cake to obtain a maximal weekly profit?

b. The supplier, unhappy with reduced sales, informs the owners that if they purchase fewer than 165 cakes weekly, the cost per cake will increase to \$7.50. Now what is the optimal sales price per cake and what is the total weekly profit?

c. Situations like this occur regularly in retail trade. Discuss the implications of reduced sales with increased total profit versus greater sales with smaller profits. For example, a drop in the number of customers means fewer sales of associated products.

Part C

13. If the cost of producing x items is given by the function $C(x)$, and the total revenue when x items are sold is $R(x)$, then the profit function is $P(x) = R(x) - C(x)$. Show that the profit function has a critical point when the marginal revenue equals the marginal cost.

14. A fuel tank is being designed to contain 200 m^3 of gasoline; however, the maximum length tank that can be safely transported to clients is 16 m long. The design of the tank calls for a cylindrical part in the middle with hemispheres at each end. If the hemispheres are twice as expensive per unit area as the cylindrical wall, then find the radius and height of the cylindrical part so that the cost of manufacturing the tank will be minimal. Give the answer correct to the nearest centimetre.

15. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 m apart, where should an object be placed on the line between the two lights so as to receive the least illumination?

16. During a cough, the diameter of the trachea decreases. The velocity, v, of air in the trachea during a cough may be modelled by the formula $v(r) = Ar^2(r_0 - r)$, where A is a constant, r is the radius of the trachea during the cough, and r_0 is the radius of the trachea in a relaxed state. Find the radius of the trachea when the velocity is the greatest, and find the associated maximum velocity of air. Note that the domain for the problem is $0 \le r \le r_0$.

Key Concepts Review

In Chapter 5, you have considered a variety of applications of derivatives.

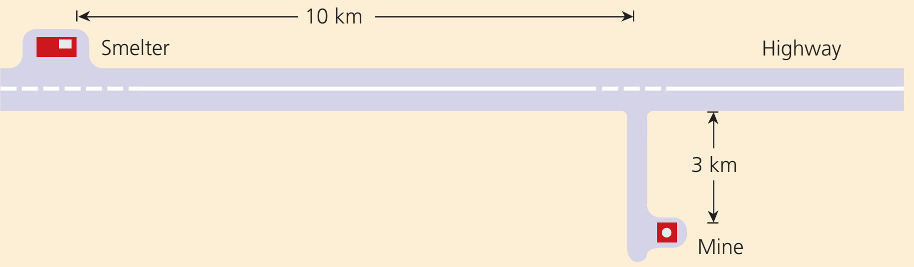
You should now be familiar with the following concepts:

- $y = f(x)$ defines y explicitly as a function of x. For example, $y = x^3 - 4x + 2$.
- An equation involving both x and y, as in $x = y^2 + 1$, is said to define y implicitly as a function of x. For implicit differentiation, we differentiate both sides of the equation with respect to x. The Chain Rule is used when differentiating terms containing y.
- The position, velocity, and acceleration functions are $s(t)$, $v(t)$, and $a(t)$ respectively, where $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$
- The algorithm for maximum and minimum values
- Derivatives in the social sciences that involve cost, revenue, and profit
- Related-rate problems
- Optimization problems. Remember that you must first create a function to analyze and that restrictions in the domain may be crucial.

Career Link wrap-up investigate and apply

CHAPTER 5: MAXIMIZING PROFITS

A construction company has been offered a build-operate contract for $7.8 million to construct and operate a trucking route for five years to transport ore from a mine site to a smelter. The smelter is located on a major highway and the mine is 3 km into the bush off the road.



Construction (capital) costs are estimated as follows:

- Upgrade to the highway (i.e., repaving) will be $200 000/km.
- New gravel road from the mine to the highway will be $500 000/km.

Operating conditions are as follows:

- There will be 100 return trips each day for 300 days a year for each of the five years the mine will be open.
- Operating costs on the gravel road will be $65/h and average speed will be 40 km/h.
- Operating costs on the highway will be $50/h and average speed will be 70 km/h.

Use calculus to determine if the company will accept the contract and the distances of the paved and gravel road sections producing optimum conditions (maximum profit). What is the maximum profit? Do not consider the time value of money in your calculations.

Review Exercise

1. Find $\frac{dy}{dx}$ for each of the following:

   a. $x^3 + y^5 = 6$

   b. $\frac{1}{x^2} + \frac{1}{y^2} = 5$

   c. $y^3 = \frac{x-1}{x+1}$

   d. $x^2y^{-3} + 3 = y$

   e. $y^7 - 2x^7y^5 = 10$

   f. $x^{\frac{2}{5}} + y^{\frac{3}{5}} = x$

2. Find $\frac{dy}{dx}$ at the indicated point.

   a. $x^3 + y^3 = 18xy$ at $(8, 4)$

   b. $(x^2 + y^2)^2 = 4x^2y$ at $(1,1)$

3. Find the slopes of the lines tangent to the graph of $x^{-2}y^6 + 2y^{-2} - 6 = 0$ at the point $(0.5, 1)$ and at the point $(0.5, -1)$.

4. Find f' and f'', if $f(x) = x^4 - \frac{1}{x^4}$.

5. For $y = x^9 - 7x^3 + 2$, find $\frac{d^2y}{dx^2}$.

6. For the relation $3x^2 - y^2 = 7$, show that $y'' = -\frac{21}{y^3}$.

7. Find the velocity and acceleration of an object that moves along a straight line in such a way that its coordinate $s(t)$ is $s(t) = t^2 + (2t - 3)^{\frac{1}{2}}$.

8. Find the velocity and acceleration as functions of time, t, for $s(t) = t - 7 + \frac{5}{t}$, $t \neq 0$.

9. A pellet is shot into the air. Its position above the ground at any time, t, is defined by $s(t) = 45t - 5t^2$ metres. For what values of t seconds, $t \geq 0$, is the upward velocity of the pellet positive? Zero? Negative? Draw a graph to represent the velocity of the pellet.

10. Determine the maximum and minimum of each function on the given interval.

    a. $f(x) = 2x^3 - 9x^2$, $-2 \leq x \leq 4$

    b. $f(x) = 12x - x^3$, $-3 \leq x \leq 5$

    c. $f(x) = 2x + \frac{18}{x}$, $1 \leq x \leq 5$

11. A motorist starts braking when she sees a stop sign. After t seconds, the distance (in metres) from the front of her car to the sign is $s(t) = 62 - 16t + t^2$.

    a. How far was the front of the car from the sign when she started braking?

    b. Does the car go beyond the stop sign before stopping?

    c. Explain why it is unlikely that the car would hit another vehicle that is travelling perpendicular to the motorist's road when her car first comes to a stop at the intersection.

12. Find the equation of the tangent to the graph $y^3 - 3xy - 5 = 0$ at the point $(2, -1)$.

13. The position function of an object that moves in a straight line is $s(t) = 1 + 2t - \frac{8}{t^2 + 1}$, $0 \le t \le 2$. Find the maximum and minimum velocities of the object over the given time interval.

14. Suppose that the cost (in dollars) of manufacturing x items is approximated by $C(x) = 625 + 15x + 0.01x^2$, for $1 \le x \le 500$. The unit cost (the cost of manufacturing one item) would then be $U(x) = \frac{C(x)}{x}$. How many items should be manufactured in order to ensure that the unit cost is minimized?

15. For each of the following cost functions, find, in dollars,

    a. the cost of producing 400 items.

    b. the average cost of each of the first 400 items produced.

    c. the marginal cost when $x = 400$, as well as the cost of producing the 401st item.

       i) $C(x) = 3x + 1000$

       ii) $C(x) = 0.004x^2 + 40x + 8000$

       iii) $C(x) = \sqrt{x} + 5000$

       iv) $C(x) = 100x^{-\frac{1}{2}} + 5x + 700$

16. Find the production level that minimizes the average cost per unit for the cost function $C(x) = 0.004x^2 + 40x + 16\,000$. Show that it is a minimum by using a graphing calculator to sketch the graph of the average cost function.

17. a. The position of an object moving along a straight line is described by the function $s(t) = 3t^2 - 10$ for $t \ge 0$. Is the object moving away from or towards its starting position when $t = 3$?

    b. Repeat the problem using $s(t) = -t^3 + 4t^2 - 10$ for $t \ge 0$.

18. Sand is being poured onto a conical pile at the rate of 9 m^3/h. Friction forces in the sand are such that the slope of the sides of the conical pile is always $\frac{2}{3}$.
    a. How fast is the altitude increasing when the radius of the base of the pile is 6 m?
    b. How fast is the radius of the base increasing when the height of the pile is 10 m?

19. Digging in his backyard, Dennis accidentally breaks a pipe attached to his water-sprinkling system. Water bubbles up at a rate of 1 cm^3/s, forming a circular pond of depth 0.5 cm in his yard. How quickly is the surface area of the pond covering his lawn?

20. The surface area of a cube is changing at a rate of 8 cm^2/s. How fast is the volume changing when the surface area is 60 cm^2?

21. A coffee filter has the shape of an inverted cone. Water drains out of the filter at a rate of 10 cm^3/min. When the depth of water in the cone is 8 cm, the depth is decreasing at 2 cm/min. What is the ratio of the height of the cone to its radius?

22. A floodlight that is 15 m away and at ground level illuminates a building. A man 2 m tall walks away from the light directly towards the building at 2 m/s. Is the length of his shadow on the building increasing or decreasing? Find the rate of change of the length of his shadow when he is 4 m from the light.

23. A particle moving along a straight line will be s cm from a fixed point at time t seconds, where $t > 0$ and $s = 27t^3 + \frac{16}{t} + 10$.
    a. Find when the velocity will be zero.
    b. Is this a maximum or a minimum velocity?
    c. Is the particle accelerating? Explain.

24. A box with a square base and no top must have a volume of 10 000 cm^3. If the smallest dimension in any direction is 5 cm, then determine the dimensions of the box that minimize the amount of material used.

25. An animal breeder wishes to create five adjacent rectangular pens, each with an area of 2400 m^2. To ensure that the pens are large enough for grazing, the minimum for either dimension must be 10 m. Find the dimensions for the pens in order to keep the amount of fencing used to a minimum.

26. You are given a piece of sheet metal that is twice as long as it is wide and the area of the sheet is 800 square decametres. Find the dimensions of the rectangular box that would contain a maximum volume if it were constructed from this piece of metal. The box will not have a lid. Give your answer correct to one decimal place.

27. A cylindrical can is to hold 500 cm^3 of apple juice. The design must take into account that the height must be between 6 and 15 cm, inclusive. How should the can be constructed so that a minimum amount of material will be used in the construction? (Assume that there will be no waste.)

28. In oil pipeline construction, the cost of pipe to go under water is 60% more than the cost of pipe used in dry land situations. A pipeline comes to a 1-km-wide river at point *A,* and it must be extended to a refinery, *R*, on the other side that is 8 km down a straight river. Find the best way to cross the river so that the total cost of the pipe is kept to a minimum. (Answer to the nearest metre.)

29. A train leaves the station at 10:00 and travels due north at a speed of 100 km/h. Another train has been heading due west at 120 km/h and reaches the same station at 11:00. At what time were the two trains closest together?

30. A store sells portable CD players for $100 each, and at this price the store sells 120 CD players every month. The owner of the store wishes to increase his profit, and he estimates that for every $2 increase in the price of CD players, one less CD player will be sold each month. If each CD player costs the store $70, at what price should the store sell the CD players to maximize profit?

31. An offshore oil well, *P*, is located in the ocean 5 km from the nearest point on the shore, *A*. A pipeline is to be built to take oil from *P* to a refinery that is 20 km along the straight shoreline from *A*. If it costs $100 000 per kilometre to lay pipe underwater and only $75 000 per kilometre to lay pipe on land, what route from the well to the refinery will be the cheapest? (Give your answer correct to the nearest metre.)

Chapter 5 Test

Achievement Category	Questions
Knowledge/Understanding	All questions
Thinking/Inquiry/Problem Solving	9, 11
Communication	5b
Application	4, 5, 6, 10

1. Find $\frac{dy}{dx}$ if $x^2 + 4xy - y^2 = 8$.

2. Find the equation of the tangent to $3x^2 + 4y^2 = 7$ at $P(-1, 1)$.

3. An object starts at rest and moves along a horizontal trail. Its position, s, in metres, after t seconds is given by $s(t) = t^3 - 9t^2 + 24t + 5$, $t \geq 0$.
   a. Find the average velocity from $t = 1$ s to $t = 6$ s.
   b. At what time(s) is the object at rest?
   c. Determine its acceleration after 5 s.
   d. Is the object moving towards or away from the origin when $t = 3$ s? Justify your answer.

These formulas may be helpful for the following questions.

Sphere $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$ **Cone** $V = \frac{1}{3}\pi r^2 h$

Cylinder $V = \pi r^2 h$, $S = 2\pi rh + 2\pi r^2$ **Circle** $C = 2\pi r$, $A = \pi r^2$

4. Assume that oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 m/s. How fast is the area of the spill increasing when the radius of the spill is 60 m?

5. The radius of a sphere is increasing at a rate of 2 m/min.
   a. Find the rate of change of the volume when the radius is 8 m.
   b. Explain why the rate of change of the volume of the sphere is not constant even though $\frac{dr}{dt}$ is constant.

6. At a certain instant, each edge of a cube is 5 cm long and the volume is increasing at a rate of 2 cm^3/min. How fast is the surface area of the cube increasing?

7. An inverted conical water tank has a radius of 10 m at the top and is 24 m high. If water flows into the tank at a rate of 20 m^3/min, how fast is the depth of the water increasing when the water is 16 m deep?

8. Determine the extreme values of the function $f(x) = \frac{x^2 - 1}{x + 2}$ on the domain $-1 \leq x \leq 3$.

9. A figure skater is directly beneath a spotlight 10 m above the ice. If she skates away from the light at a rate of 6 m/s and the spot follows her, how fast is her shadow's head moving when she is 8 m from her starting point? The skater is (almost) 1.6 m tall with her skates on.

10. A man has purchased 2000 m of used wire fencing at an auction. He and his wife want to use the fencing to create three adjacent rectangular paddocks. Find the dimensions of the paddocks so the fence encloses the largest possible area.

11. An engineer working on a new generation of computer called The Beaver is using very compact VLSI circuits. The container design for the CPU is to be determined by marketing considerations and must be a rectangular solid in shape. It must contain exactly 10 000 cm^3 of interior space, and the length must be twice the height. If the cost of the base is \$0.02/cm^2, the cost of the side walls is \$0.05/cm^2, and the cost of the upper face is \$0.10/cm^2, find the dimensions to the nearest millimetre that will keep the cost of the container to a minimum.